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Steady and transient Green's functions for anisotropic conduction in an exponentially graded solid

Hsin-Yi Kuo, Tungyang Chen *

Department of Civil Engineering, National Cheng Kung University, 1 University Road, Tainan 70101, Taiwan

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Abstract

The problem of the determination of Green's function in conduction for a rectilinearly anisotropic solid with an exponential grading along a certain direction is studied. Domains of an unbounded space and a half-space, either three-dimensional or two-dimensional, are considered. Along the boundary of the domain, homogeneous boundary conditions of the first and second kinds are imposed. We find interestingly that, under this specific type of grading, the Green's functions permit an algebraic decomposition, which will in turn greatly simplify the formulation. The method of Fourier transform is employed for the Green's function for a half-space or a half-plane. Although the derivation process is quite tedious, we show analytically that the inverse transform can be found exactly and their resulting expressions are surprisingly neat and compact. In addition, both steady-state and transient-state field solutions are considered. By taking Laplace transform with respect to the time variable, we show that the mathematical frameworks for the steady-state and transient-state Green's functions are entirely analogous. Thereby, the transient-state Green's function is readily obtained by taking Laplace inverse transform back to the time domain. These derived fundamental solutions will serve as benchmark results for modeling some inhomogeneous materials. In the absence of grading term, we have verified analytically that our solutions agree exactly with previously known Green's functions for homogeneous media.
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1. Introduction

Finding Green's functions pertaining to certain physical phenomena, possibly incorporated with suitably prescribed boundary data, is one of the fundamental subjects in mathematical physics. The knowledge of Green's functions can serve as basic ingredient to construct the fields via superpositions under various distributed sources and general boundary data. Classical fundamental solutions, such as in conduction and in

* Corresponding author. Tel.: +886 6 2757575 63121; fax: +886 6 2358542.
E-mail address: tchen@mail.ncku.edu.tw (T. Chen).

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elasticity (known as Kelvin's solution), have been known for more than one century. Existing Green's functions of other physical phenomena include anisotropic piezoelectricity, thermoelasticity and poroelasticity (Norris, 1994; Chen, 1993). Most of the existing fundamental solutions are under the condition that the material is homogeneous. Recently, there has been a growing interest on functionally graded materials, i.e. materials with spatially varying properties tailored to satisfy particular engineering applications. Related works in various applications can be found, for instance, in Reiter et al. (1997), Wang and Jasiuk (1998), Horgan and Chan (1999) and Weng (2003). For a good overview of research on functionally graded materials, the reader is referred to Hirai (1995) and Suresh (2001).

Existing Green's functions for various physical phenomena in graded media are not many. The field equations for Green's functions of graded media are in general governed by partial differential equations with position-dependent coefficients, and thus finding their explicit solutions is considerably complicated. Typical solutions of Green's functions are often expressed as series or integral forms (in the transformed space) which in general delimit their applications. For some aspects of applications it is often desirable to have closed-form expressions for the Green's functions, such as in effective medium theories and in boundary integral methods. In a recent study, Martin et al. (2002) derived the Green's function for a three-dimensional exponentially graded elastic solid, in which the Lamé constants are varying exponentially along a certain direction. They showed that the Green's function consists of two parts: one, corresponding to the Green's function of a homogeneous medium, i.e., Kelvin's solution, and the other, expressing the effect associated with the grading part. However, the latter term, involves two integrals of some elementary functions, cannot be evaluated exactly.

Motivated by this latter study, we elaborate further along this line of exploration, but focus on a simpler mathematical framework, the conduction phenomena, aiming at finding exact, closed-form expressions for the Green's functions. As in Martin et al. (2002) we assume that the material property has an exponential grading along a certain direction, but the conductivity tensor is generally anisotropic. Physically, this simulates a layered medium with a continuously varying conductivity. Domains of an unbounded space, an infinite plane, a half-space and a half-plane are considered. Along the boundaries of the semi-infinite region, homogeneous boundary conditions of the first- and second-kind are considered. For the first-kind boundary condition the temperature potential is set equal to zero along the boundary, while for the second-kind the normal component of the heat flux is taken to be zero. In addition, both steady-state and transient-state field solutions are examined. We find, remarkably, under this specific type of grading, the Green's functions permit an algebraic decomposition (2.6). This will substantially reduces the algebraic complexity through the entire analysis, and thereby enhances its general transparency. The plan of the paper is as follows. In Section 2, we derive the Green's function for an unbounded space. We show that the steady-state Green's function for an exponentially graded medium is simply that of a modified Helmholtz equation multiplied by an exponential term. Green's function for a half-space is considered in Section 3. The method of double Fourier transform is employed and the solution in the transformed space is derived exactly. After some tedious derivations, we show that the inverse transform can be found analytically and their resulting expressions are surprisingly simple. The solution forms imply that the Green's function can be constructed from the fields produced by a certain distribution of image points. In Section 4, we derive the Green's functions for a two-dimensional plane, either unbounded or half-plane. In contrast to those for a three-dimensional space the mathematical formulation is considerably simpler. We show again that the exact Green's functions can be obtained in simple, closed forms. In Section 5, we consider that the transient-state Green's functions for an infinite space, an infinite plane, a half-space and a half-plane. By taking Laplace transform with respect to the time variable, we show that the mathematical frameworks for the steady-state and transient-state Green's functions are entirely analogous. Thus, the critical step in seeking the transient-state Green's functions is to invert the (Laplace) transformed solutions back to the time domain analytically. We will show that this is indeed possible. Lastly, in the absence of grading term, we have verified analytically that our solutions recover exactly with previously known Green's functions for homogeneous media.

2. Green's function for an infinite space

We consider an unbounded rectilinearly anisotropic solid which is spatially graded along a certain direction with respect to $\mathbf{x}$. Here $\mathbf{x} = (x_1, x_2, x_3)$ is a position vector corresponding to a Cartesian coordinate. We denote the conductivity tensor of the material by $\tilde{k}_{ij}(\mathbf{x})$, in which $\tilde{k}_{ij} = \tilde{k}_{ji}$. The field equilibrium equation under steady-state condition subject to a point source at $\mathbf{x}'$ is of the form

$$\frac{\partial}{\partial x_i}\left(\tilde{k}_{ij}(\mathbf{x})\frac{\partial G}{\partial x_j}\right) = -\delta(\mathbf{x}-\mathbf{x}'), \quad i,j = 1,2,3, \tag{2.1}$$

in which $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function. It is noted that sums over repeated indices are implied throughout the paper. Here the Green's function G is the temperature potential at the point $\mathbf{x}$ due to a point source applied at $\mathbf{x}'$. We now consider a particular graded material in which the conductivity tensor varies exponentially along a certain direction $\boldsymbol{\beta}$,

$$\tilde{k}_{ij}(\mathbf{x}) = k_{ij}\exp(2\boldsymbol{\beta}\cdot\mathbf{x}). \tag{2.2}$$

Here k_{ij} is a constant second-order tensor and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ is a given constant vector. As in Martin et al. (2002), the factor of 2 in the exponent is introduced here for later algebraic convenience. The assumption of an exponentially varying property is common in the engineering literature in modeling functionally graded materials. Relevant works include Giannakopoulos and Suresh (1997), Martin et al. (2002) and the references contained therein. Since

$$\frac{\partial}{\partial x_i}\tilde{k}_{ij}(\mathbf{x}) = 2\beta_i k_{ij}\exp(2\boldsymbol{\beta}\cdot\mathbf{x}) = 2\beta_i\tilde{k}_{ij}(\mathbf{x}), \tag{2.3}$$

equation (2.1) can be recast as

$$k_{ij}\frac{\partial^2 G}{\partial x_i \partial x_j} + 2\beta_i k_{ij}\frac{\partial G}{\partial x_j} = -\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))\delta(\mathbf{x}-\mathbf{x}'). \tag{2.4}$$

Here in deriving (2.4) we have employed the following identity of the Dirac delta function (Sneddon, 1972, p.486)

$$f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') = f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}'). \tag{2.5}$$

Equation (2.4) is the governing equation for Green's function for a material with an exponentially graded conductivity tensor (2.2). In the context of elasticity, under a similar exponentially grading assumption for the Lamé constants, the subject was first explored by Martin et al. (2002). They found that, apart from a common factor of $\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))$, the Green's tensor for a three-dimensional elastic isotropic medium consists of two parts: one singular part and the other grading part. The singular term is exactly the Kelvin solution, i.e. the Green's tensor for a homogeneous medium. While the grading part cannot be resolved analytically, but consists of a sum of integral of modified Bessel functions and double integrals of finite regions of elementary functions. In contrast, here in the context of conductivity we shall demonstrate that the Green's function admits simple closed-form expression.

To show this, we first claim that the Green's function G follows the decomposition

$$G(\mathbf{x},\mathbf{x}') = \exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))G^e(\mathbf{x},\mathbf{x}'). \tag{2.6}$$

By substituting (2.6) into (2.4) we see that the exponential term was cancelled out identically and the function G^e is simply governed by

$$k_{ij}\frac{\partial^2 G^e}{\partial x_i \partial x_j} - k_{ij}\beta_i\beta_j G^e + \delta(\mathbf{x}-\mathbf{x}') = 0. \tag{2.7}$$

Remarkably, Equation (2.7) is exactly *the Green's function for a modified Helmholtz operator* associated with a general anisotropic tensor k_{ij}. This observation implies that the Green's function for an exponentially graded medium is simply that of a modified Helmholtz equation multiplied by an exponential term (2.6).

To proceed for the solution for G^e in (2.7), we introduce an affine coordinate transformation (Milton, 2002)

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{2.8}$$

where the transformation matrix $\mathbf{A}$ does not depend on $\mathbf{x}$. Using the chain rule of differentiation, we have

$$\frac{\partial^2 G^e}{\partial x_i \partial x_j} = A_{ki} A_{lj} \frac{\partial^2 G^e}{\partial y_k \partial y_l}. \tag{2.9}$$

Further, we note that (DeSanto, 1992)

$$\delta(\mathbf{x} - \mathbf{x}') = |J|\, \delta(\mathbf{y} - \mathbf{y}'), \tag{2.10}$$

where $|J|$ is the absolute value of the Jacobian given by

$$|J| \equiv \left| \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \right|. \tag{2.11}$$

Accordingly, Eq. (2.7) becomes

$$A_{ki} k_{ij} A_{lj} \frac{\partial^2 G^e}{\partial y_k \partial y_l} - k_{ij} \beta_i \beta_j G^e + |J|\, \delta(\mathbf{y} - \mathbf{y}') = 0. \tag{2.12}$$

Now if one takes

$$\mathbf{A} = \mathbf{k}^{-1/2}, \tag{2.13}$$

Eq. (2.12) becomes

$$\kappa \nabla^2 G^e - \kappa \lambda^2 G^e + \delta(\mathbf{y} - \mathbf{y}') = 0, \tag{2.14}$$

where

$$\kappa = |J|^{-1} = \det \mathbf{k}^{1/2}, \quad \lambda^2 = \boldsymbol{\beta}^{\mathrm{T}} \mathbf{k} \boldsymbol{\beta}, \quad \nabla^2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}. \tag{2.15}$$

Here the superscript T indicates a matrix transpose. Now (2.12) is the standard form of the Green's function for a modified Helmholtz operator in the $\mathbf{y}$-space, the solution of which can be readily found in the literature (see for instance, Arfken and Weber, 2001, p. 554)

$$G^e = \frac{1}{4\pi\kappa\, |\mathbf{y} - \mathbf{y}'|} \exp(-\lambda\, |\mathbf{y} - \mathbf{y}'|). \tag{2.16}$$

Since

$$\begin{aligned} |\mathbf{y} - \mathbf{y}'| &= [(\mathbf{y} - \mathbf{y}')^{\mathrm{T}} (\mathbf{y} - \mathbf{y}')]^{1/2} = [(\mathbf{x} - \mathbf{x}')^{\mathrm{T}} (\mathbf{k}^{\mathrm{T}})^{-1/2} \mathbf{k}^{-1/2} (\mathbf{x} - \mathbf{x}')]^{1/2} \\ &= [(\mathbf{x} - \mathbf{x}')^{\mathrm{T}} \mathbf{k}^{-1} (\mathbf{x} - \mathbf{x}')]^{1/2}, \end{aligned} \tag{2.17}$$

we can write the function G^e (2.16) in terms of the position vector $\mathbf{x}$ in the form

$$G^e = \frac{1}{4\pi\kappa R} \exp(-\lambda R), \quad \text{with } R \equiv [(\mathbf{x} - \mathbf{x}')^{\mathrm{T}} \mathbf{k}^{-1} (\mathbf{x} - \mathbf{x}')]^{1/2}. \tag{2.18}$$

Upon substitution of (2.18) back into (2.6), the steady-state Green's function of conduction for an exponentially graded medium takes the form

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\kappa R} \exp(-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}') - \lambda R). \tag{2.19}$$

The first term characterizes the Green's function corresponding to a homogeneous conductivity **k**, while the second part is a correction factor associated with the exponentially grading term. Of course, when $\boldsymbol{\beta} = \mathbf{0}$ the inhomogeneous part takes the value of unity and the known solution for a homogeneous medium is recovered (Chang et al., 1973, Eq. (18)). We mention that the conduction Green's function for the same type of exponential grading has recently been considered by Gray et al. (2003) with the condition that the medium is isotropic. We have checked that our result (2.19) agrees with their solution under the isotropic condition. In closing this section, we remark that the affine transformation could serve as a convenient tool in resolving potential problems of various kinds. Milton's (2002) book contains some of the illustrations. Another famous example is the procedures proposed for Saint-Venant's torsion of anisotropic shafts (Sokolnikoff, 1956). We have made some substantial progress recently on related subjects based on this transformation. Detailed results will appear in a future publication.

3. Green's function for a half-space

We now derive the steady state Green's function for a half-space. As in Section 2, we assume that the conductivity tensor **k** is generally anisotropic and that the grading direction $\boldsymbol{\beta}$ could be arbitrary. Thereby, without loss of any generality, we can consider that the half-space occupies the region $0 \leqslant x_1 < \infty$. Along the boundary of the domain, $x_1 = 0$, we suppose that homogeneous types of boundary conditions are imposed. For the first-kind boundary condition the temperature potential is set equal to zero along the boundary, while for the second-kind the normal component of heat flux is taken to be zero. That is,

$$G|_{x_1=0} = 0, \quad \text{or} \quad k_{1j}\frac{\partial G}{\partial x_j}\bigg|_{x_1=0} = 0. \tag{3.1}$$

By the decomposition (2.6), the governing field for G^e still follows (2.7), but the boundary conditions (3.1) are, respectively, transformed to

$$G^e|_{x_1=0} = 0, \quad \text{or} \quad k_{1j}\frac{\partial G^e}{\partial x_j} - k_{1j}\beta_j G^e\bigg|_{x_1=0} = 0. \tag{3.2}$$

The above two equations demonstrate that a second-kind of homogeneous boundary condition in graded media is analogous to that of a convective type (a third-type) of boundary condition in homogeneous media. To proceed, we first expand the Dirac delta function in (2.7) via double Fourier transforms with respect to x_2 and x_3, [1]

[1] Note that the Fourier transform of a function $f(x)$ with respect to the variable x and its inverse transform are defined here as

$$\tilde{f}(\xi) \equiv \mathscr{F}(f(x)) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x,$$
$$f(x) \equiv \mathscr{F}^{-1}(\tilde{f}(\xi)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(\xi)\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}\xi.$$

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi^2}\delta(x_1-x_1')\int_{-\infty}^{\infty}\exp[\mathrm{i}\xi_2(x_2-x_2')]\,\mathrm{d}\xi_2\int_{-\infty}^{\infty}\exp[\mathrm{i}\xi_3(x_3-x_3')]\,\mathrm{d}\xi_3, \tag{3.3}$$

and assume a similar expansion for the function G^e as

$$G^e(x_1,x_2,x_3) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}^e(x_1,\xi_2,\xi_3)\exp[\mathrm{i}\xi_2(x_2-x_2')]\exp[\mathrm{i}\xi_3(x_3-x_3')]\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3. \tag{3.4}$$

Note that $\tilde{G}^e(x_1, \xi_2, \xi_3)$ is exactly the Fourier transform of G^e with respect to the x_2 and x_3 variables. A direct substitution of (3.3) and (3.4) into (2.7) will show that the function $\tilde{G}^e$ is governed by

$$k_{11}\frac{\mathrm{d}^2\tilde{G}^e}{\mathrm{d}x_1^2} + 2\mathrm{i}(k_{12}\xi_2 + k_{13}\xi_3)\frac{\mathrm{d}\tilde{G}^e}{\mathrm{d}x_1} - \hat{\mu}^2\tilde{G}^e = -\delta(x_1-x_1'), \tag{3.5}$$

and the boundary condition (3.2) becomes

$$\tilde{G}^e|_{x_1=0} = 0, \quad \text{or} \quad k_{11}\frac{\mathrm{d}\tilde{G}^e}{\mathrm{d}x_1} + [\mathrm{i}(k_{12}\xi_2 + k_{13}\xi_3) - k_{1j}\beta_j]\tilde{G}^e\bigg|_{x_1=0} = 0, \tag{3.6}$$

where we have set

$$\hat{\mu}^2 \equiv \boldsymbol{\xi}^{\mathrm{T}}\mathbf{k}_{(11)}\boldsymbol{\xi} + \lambda^2, \quad \boldsymbol{\xi} \equiv (\xi_2, \xi_3)^{\mathrm{T}}, \tag{3.7}$$

and $\mathbf{k}_{(ij)}$, $i,j = 1,\ldots,3$, is the (2×2) submatrix of $\mathbf{k} \equiv (k_{ij})$ by deleting the row and column containing the element k_{ij}. To resolve (3.5), we use the substitution (Lanczos, 1961, p. 359)

$$\tilde{G}^e(x_1,\xi_2,\xi_3) = \tilde{G}^w(x_1,\xi_2,\xi_3)/w(x_1), \tag{3.8}$$

in which

$$w(x_1) = \exp(\mathrm{i}x_1(k_{12}\xi_2 + k_{13}\xi_3)/k_{11}). \tag{3.9}$$

This transformation has the conspicuous properties that it will eliminate the first derivative term and will also transform the differential operator into a self-adjoint form. Specifically, some algebraic manipulations of the system, (3.5) and (3.6), will yield

$$\frac{\mathrm{d}^2\tilde{G}^w}{\mathrm{d}x_1^2} - \mu^2\tilde{G}^w = -\delta(x_1-x_1')w(x_1')/k_{11}, \tag{3.10}$$

together with

$$\tilde{G}^w|_{x_1=0} = 0, \quad \text{or} \quad -\frac{\mathrm{d}\tilde{G}^w}{\mathrm{d}x_1} + p\tilde{G}^w\bigg|_{x_1=0} = 0, \quad \text{with } p \equiv k_{1j}\beta_j/k_{11}, \tag{3.11}$$

where

$$\mu^2 \equiv (\boldsymbol{\xi}^{\mathrm{T}}\mathbf{m}\boldsymbol{\xi} + \lambda^2 k_{11})/k_{11}^2, \quad \text{and} \quad \mathbf{m} = \begin{pmatrix} m_{33} & m_{23} \\ m_{23} & m_{22} \end{pmatrix}, \quad \text{with } m_{ij} = \det\mathbf{k}_{(ij)}. \tag{3.12}$$

Equation (3.10) can be solved using the usual solution procedures for Green's function (Hildebrand, 1965, p.228). For completeness, we outline its solution procedures in Appendix B. Specifically, the function $\tilde{G}^w$ can be resolved as

$$\tilde{G}^w(x_1,\xi_2,\xi_3) = \frac{w(x_1')}{2\mu k_{11}}[\exp(-\mu(x_{1>} - x_{1<})) - \exp(-\mu(x_{1>} + x_{1<}))], \tag{3.13}$$

for the first-kind boundary condition $(3.11)_1$, and as

$$\widetilde{G}^{w}(x_1,\xi_2,\xi_3)=\frac{w(x_1')}{2\mu k_{11}}\left[\exp(-\mu(x_{1>}-x_{1<}))+\frac{\mu-p}{\mu+p}\exp(-\mu(x_{1>}+x_{1<}))\right], \tag{3.14}$$

for the second-kind condition $(3.11)_2$, where we have defined

$$x_{1>}\equiv\max(x_1,x_1'),\quad x_{1<}\equiv\min(x_1,x_1'). \tag{3.15}$$

Now substitution of (3.13) or (3.14) together with (3.8) back into (3.4) will lead to an integral form for the solution of G^e, which is indeed the Fourier inversion of $\widetilde{G}^{e}$. In general, finding the closed-form expressions for a double Fourier inverse transform may not be a simple task. Remarkably, despite their cumbersome appearances, after a great deal of effort, we are able to show analytically in Appendices C and D that they can be exactly integrated. The derived solutions for G^e are given in (C.16) and (D.7). Consequently, a substitution of G^e back to (2.6) will lead to our desired half-space Green's function G,

$$G(\mathbf{x},\mathbf{x}')=\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))\left(\frac{\exp(-\lambda R)}{4\pi\kappa R}-\frac{\exp(-\lambda R_i)}{4\pi\kappa R_i}\right), \tag{3.16}$$

for the first-kind boundary condition, and

$$G(\mathbf{x},\mathbf{x}')=\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))\left(\frac{\exp(-\lambda R)}{4\pi\kappa R}+\frac{\exp(-\lambda R_i)}{4\pi\kappa R_i}+\int_0^\infty h(\zeta)\frac{\exp(-\lambda R_\zeta)}{4\pi\kappa R_\zeta}\mathrm{d}\zeta\right), \tag{3.17}$$

corresponding to the second-kind boundary condition, where R was given in (2.18), $h(\zeta)$ is defined in (D.5) and

$$R_i=\sqrt{R^2+\frac{4x_1x_1'}{k_{11}}},\quad R_\zeta=\sqrt{R^2+\frac{(2x_1+\zeta)(2x_1'+\zeta)}{k_{11}}}. \tag{3.18}$$

It should be noted that, for brevity of expressions, the variables R_i and R_ζ, defined above, have been rewritten from (C.12), (C.15) and (D.7) by simple addition and substraction. For (3.16), we observe the first term on the right, after multiplication, is exactly the Green's function for a graded infinite space, see (2.19). While the second term on the right also represents the Green's function for a graded unbounded space but the image point $\mathbf{x}_1'$ is shifted in a specific manner (see (C.12) and (C.15)). For (3.17), the first two terms on the right again can be interpreted as superposition of imaged points. The third term, which involves an integral expression, is however an additional term due to the grading effect. To further elaborate on this point, we introduce a new dummy variable ζ' by letting $\zeta'=-x_1'-\zeta$. This will allow us to rewrite (3.17) in the alternative form

$$G(\mathbf{x},\mathbf{x}')=\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))\left(\frac{\exp(-\lambda R)}{4\pi\kappa R}+\frac{\exp(-\lambda R_i)}{4\pi\kappa R_i}+\int_{-\infty}^{-x_1'} h(\zeta')\frac{\exp(-\lambda R_{\zeta'})}{4\pi\kappa R_{\zeta'}}\mathrm{d}\zeta'\right), \tag{3.19}$$

where

$$h(\zeta')=-2p\ \exp(p(x_1'+\zeta')),\quad R_{\zeta'}=\sqrt{R^2+\frac{(2x_1-x_1'-\zeta')(x_1'-\zeta')}{k_{11}}}. \tag{3.20}$$

We note that this expression (3.19) is formally similar to the Green's function of a harmonic operator for a half-space with a third-type boundary condition (Greenberg, 1971, p. 86), which can be interpreted as a continuous variation of imaged singularities. This in fact reflects our previous observation that a second-kind of boundary condition in this exponentially graded media is mathematically analogous to that of a convective type of boundary condition in homogeneous media.

Lastly, if one sets $\boldsymbol{\beta} = \mathbf{0}$, it can be readily seen that $h(\zeta)$ vanishes identically. Further, it can be verified analytically that (3.16) exactly reduces to the corresponding known Green's function for a homogeneous half-space (Chang, 1977, Eq. (5.4)).

4. Two-dimensional medium

4.1. Green's function for an infinite plane

Our previous formulation could also apply to a two-dimensional plane. Of course, the mathematical derivations will be considerably simpler than those of the three-dimensional space. Here in this section we shall examine the corresponding Green's function for an exponentially graded plane. The previous framework for three-dimensional Green's function in Section 2 remains valid, but the indices i, j now take the values of 1 and 2. For plane problems, $\mathbf{k}$ is a (2×2) matrix and the position vector is defined by $\mathbf{x} = (x_1, x_2)$. As in Section 2, the field equation for G^e is still governed by (2.14), with the definition

$$\boldsymbol{\beta} = (\beta_1, \beta_2), \quad \nabla^2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}. \tag{4.1}$$

The solution of Eq. (2.14) for a two-dimensional plane can be found from Arfken and Weber (2001) as

$$G^e(\mathbf{y}, \mathbf{y}') = \frac{1}{2\pi\kappa} K_0(\lambda\,|\mathbf{y} - \mathbf{y}'|), \tag{4.2}$$

which can be readily transformed back to physical space $\mathbf{x}$ as

$$G^e = \frac{1}{2\pi\kappa} K_0(\lambda R), \tag{4.3}$$

where K_0 is the modified Bessel function of the second kind, of order zero. Note that the variable R for a two-dimensional plane has the same formal expression as in (2.18) for a three-dimensional space. But they are intrinsically different as here $\mathbf{k}$ is a (2×2) matrix and $\mathbf{x} = (x_1, x_2)$. Accordingly the Green's function for an exponentially graded material in a two-dimension medium then becomes

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi\kappa} K_0(\lambda R) \exp(-\boldsymbol{\beta} \cdot (\mathbf{x} + \mathbf{x}')). \tag{4.4}$$

4.2. Green's function for a half-plane

We now derive the Green's function for an exponentially graded half-plane. Without loss of any generality we assume that the half plane occupies the region $0 \leqslant x_1 < \infty$. Along the boundary of the domain, $x_1 = 0$, two types of homogeneous boundary conditions are considered. Since the derivation steps are entirely parallel to that of three dimensional case, only the main difference will be illustrated.

Following the decomposition (2.6), it can be verified that the decomposed Green's function $G^e(x_1, x_2)$ is governed by (2.7) together with the boundary conditions (3.2). Analogous to (3.3) and (3.4), we can write

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x}') &= \frac{1}{2\pi} \delta(x_1 - x_1') \int_{-\infty}^{\infty} \exp[\mathrm{i}\xi_2(x_2 - x_2')]\mathrm{d}\xi_2, \\ G^e(x_1, x_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{G}^e(x_1, \xi_2) \exp[\mathrm{i}\xi_2(x_2 - x_2')]\mathrm{d}\xi_2. \end{aligned} \tag{4.5}$$

This will allow us to show that the function $\widetilde{G}^e$ is described as in (3.5) with the condition (3.6). But now $\hat{\mu}$ is simplified as

$$\hat{\mu}^2 \equiv \lambda^2 + k_{22}\xi_2^2, \tag{4.6}$$

It should be mentioned when referring to previous equations for a three-dimensional space, the terms associated with x_3 or ξ_3 should be taken out. Further, we can introduce a change of variable (3.8), omitting ξ_3, by setting

$$w(x_1) = \exp(\mathrm{i}k_{12}\xi_2 x_1/k_{11}). \tag{4.7}$$

This will give exactly (3.10) and (3.11), and thus the resulting solutions for $\widetilde{G}^w$ in (3.13) and (3.14) remains the same, with now the new definition

$$\mu^2 = (\xi_2^2 \det \mathbf{k} + \lambda^2 k_{11})/k_{11}^2. \tag{4.8}$$

Now substitution of (3.13) or (3.14) together with (3.8) with (4.7) back into (4.5) will lead to an integral form for the solution of G^e. Note that the major difference between those for 3D and 2D configurations is that for a three-dimensional space, G^e is the double Fourier inversion with respect to ξ_2 and ξ_3, while for a 2D configuration, G^e is the Fourier inversion with respect to one single variable ξ_2. Thus the mathematical formulation for the latter problem is considerably simpler. Similar to the steps outlined in Appendices C and D, we find that the half-plane Green's functions are exactly derived as

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi\kappa}\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))[K_0(\lambda R) - K_0(\lambda R_i)], \tag{4.9}$$

for the first-kind boundary condition, and

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi\kappa}\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))\left[K_0(\lambda R) + K_0(\lambda R_i) + \int_0^\infty h(\zeta)K_0(\lambda R_\zeta)\mathrm{d}\zeta\right], \tag{4.10}$$

for the second-kind boundary condition, where $h(\zeta)$ is defined in (D.5) and R, R_i and R_ζ for a 2D configuration are given by

$$\begin{aligned}
R^2 &= \frac{(x_{1>} - x_{1<})^2}{k_{11}} + \frac{k_{11}}{\det \mathbf{k}}\left[-\frac{k_{12}}{k_{11}}(x_1 - x_1') + (x_2 - x_2')\right]^2, \\
R_i^2 &= \frac{(x_{1>} + x_{1<})^2}{k_{11}} + \frac{k_{11}}{\det \mathbf{k}}\left[-\frac{k_{12}}{k_{11}}(x_1 - x_1') + (x_2 - x_2')\right]^2, \\
R_\zeta^2 &= \frac{(x_{1>} + x_{1<} + \zeta)^2}{k_{11}} + \frac{k_{11}}{\det \mathbf{k}}\left[-\frac{k_{12}}{k_{11}}(x_1 - x_1') + (x_2 - x_2')\right]^2.
\end{aligned} \tag{4.11}$$

Note that the relation (3.18) remains valid for (4.11).

5. Transient Green's functions

5.1. An infinite space

We now examine the transient Green's function of conduction for an infinite space. In a fixed rectangular coordinate $\mathbf{x}$, the Green's function $G(\mathbf{x},\mathbf{x}';\, t,t')$ of conduction at transient state has the form

$$\frac{\partial}{\partial x_i}\left(\tilde{k}_{ij}(\mathbf{x})\frac{\partial G}{\partial x_j}\right) - \frac{1}{\hat{\alpha}(\mathbf{x})}\frac{\partial G}{\partial t} = -\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{5.1}$$

Here $\hat{\alpha}(\mathbf{x})$ is a material constant depending on specific heat and mass density. We consider that $\hat{\alpha}(\mathbf{x})$ follows a similar functional variation as the conductivity $\tilde{k}_{ij}(\mathbf{x})$ in (2.2), namely

$$\frac{1}{\hat{\alpha}(\mathbf{x})} = \frac{1}{\alpha}\exp(2\boldsymbol{\beta}\cdot\mathbf{x}). \tag{5.2}$$

Indeed the assumption that $\tilde{k}_{ij}$ and $\hat{\alpha}$ possess the same grading is somewhat restricted and may not be realistic in practice. But we shall demonstrate that, under this prerequisite, the transient Green's functions for various cases can be resolved analytically. This will serve as benchmark solutions among very few existing solutions of this kind. Back to (5.1), it will be seen that the Green's function G can still be separated as in (2.6)

$$G(\mathbf{x},\mathbf{x}';t,t') = \exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}'))G^e(\mathbf{x},\mathbf{x}';t,t'). \tag{5.3}$$

Direct substitution of (5.3) into (5.1) will find that the exponential term can be cancelled out identically and the field of G^e is governed by

$$k_{ij}\frac{\partial^2 G^e}{\partial x_i \partial x_j} - k_{ij}\beta_i\beta_j G^e - \frac{1}{\alpha}\frac{\partial G^e}{\partial t} + \delta(\mathbf{x}-\mathbf{x}')\delta(t-t') = 0. \tag{5.4}$$

To proceed, we make use of the Laplace transform

$$\mathscr{L}\{f(t)\} \equiv \int_0^\infty e^{-st}f(t)\mathrm{d}t = \overline{f}(s), \tag{5.5}$$

where the overbar of $\overline{f}(s)$ denotes the Laplace transform of $f(t)$. The Laplace transform of (5.4) with respect to t will give

$$k_{ij}\frac{\partial^2 (e^{st'}\overline{G^e})}{\partial x_i \partial x_j} - \left(k_{ij}\beta_i\beta_j + \frac{s}{\alpha}\right)(e^{st'}\overline{G^e}) + \delta(\mathbf{x}-\mathbf{x}') = 0, \tag{5.6}$$

where we have employed the vanishing of initial condition and, for clarity, we have multiplied the function $\exp(st')$ throughout. We note that in comparison of (2.7) for steady-state conditions, the two sets of governing fields, (5.6) and (2.7), are entirely analogous by setting

$$G^e \leftrightarrow e^{st'}\overline{G^e}, \quad k_{ij}\beta_i\beta_j \leftrightarrow k_{ij}\beta_i\beta_j + \frac{s}{\alpha}. \tag{5.7}$$

The simple correspondence of (5.7) suggests that we can directly write down the solution of (5.6) via (2.18) as

$$e^{st'}\overline{G^e} = \frac{1}{4\pi\kappa R}\exp(-\tilde{\lambda}R), \tag{5.8}$$

where

$$\tilde{\lambda}^2 = k_{ij}\beta_i\beta_j + \frac{s}{\alpha}. \tag{5.9}$$

Making use of the identity (A.1), the inverse Laplace transform of (5.8) follows

$$G^e = \frac{H(t-t')\alpha}{\kappa[4\pi\alpha(t-t')]^{\frac{3}{2}}}\exp\left(-\frac{R^2}{4\alpha(t-t')} - \alpha\lambda^2(t-t')\right), \tag{5.10}$$

where $H(t-t')$ is the Heaviside unit step function. Thus, from (5.3), we have

$$G(\mathbf{x},\mathbf{x}';t,t') = \frac{H(t-t')\alpha}{\kappa[4\pi\alpha(t-t')]^{\frac{3}{2}}}\exp\left(-\frac{R^2}{4\alpha(t-t')}\right) \cdot \exp[-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')]. \tag{5.11}$$

Equation (5.11) again implies that the Green's function for an exponentially graded material can be regarded as a product of two parts. The homogeneous part is exactly the Green's function for a homogeneous solid with conductivity tensor k_{ij}

$$G = \frac{H(t-t')\alpha}{\kappa[4\pi\alpha(t-t')]^{\frac{3}{2}}}\exp\left(-\frac{R^2}{4\alpha(t-t')}\right). \tag{5.12}$$

While the grading part is due to the grading (inhomogeneous) effect of conductivity which can be viewed as a correction term. We mention that (5.12) also agrees with the known Green's function for a homogeneous solid with conductivity $\mathbf{k}$ (Chang et al., 1973, Eq. (16)).

5.2. A half-space

We now examine the transient-state Green's function for a half-space. It is seen that (5.1) to (5.6) remain valid and the boundary conditions, via Laplace's transform, are

$$\overline{G^e}|_{x_1=0} = 0, \quad \text{or} \quad k_{1j}\frac{\partial\overline{G^e}}{\partial x_j} - k_{1j}\beta_j\overline{G^e}\bigg|_{x_1=0} = 0. \tag{5.13}$$

The algebraic correspondence of (5.7) suggests that the solutions in the transformed domain should be

$$e^{st'}\overline{G^e} = \frac{\exp(-\tilde{\lambda}R)}{4\pi\kappa R} - \frac{\exp(-\tilde{\lambda}R_i)}{4\pi\kappa R_i}, \tag{5.14}$$

for $(5.13)_1$ and

$$e^{st'}\overline{G^e} = \frac{\exp(-\tilde{\lambda}R)}{4\pi\kappa R} + \frac{\exp(-\tilde{\lambda}R_i)}{4\pi\kappa R_i} + \int_0^\infty h(\zeta)\frac{\exp(-\tilde{\lambda}R_\zeta)}{4\pi\kappa R_\zeta}\mathrm{d}\zeta, \tag{5.15}$$

for $(5.13)_2$. Now making use of the identity (A.1), the inverse Laplace transform of (5.14) and (5.15) can be exactly found. These, in conjunction with (5.3), will give the transient Green's function

$$G(\mathbf{x},\mathbf{x}';t,t') = \frac{H(t-t')\alpha}{\kappa[4\pi\alpha(t-t')]^{\frac{3}{2}}}\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')) \times \left(\exp\left(-\frac{R^2}{4\alpha(t-t')}\right) - \exp\left(-\frac{R_i^2}{4\alpha(t-t')}\right)\right), \tag{5.16}$$

for the first-kind boundary condition, and

$$G(\mathbf{x},\mathbf{x}';t,t') = \frac{H(t-t')\alpha}{\kappa[4\pi\alpha(t-t')]^{\frac{3}{2}}}\exp[-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')] \times \left[\exp\left(-\frac{R^2}{4\alpha(t-t')}\right) + \exp\left(-\frac{R_i^2}{4\alpha(t-t')}\right) + \int_0^\infty h(\zeta)\exp\left(-\frac{R_\zeta^2}{4\alpha(t-t')}\right)\mathrm{d}\zeta\right]. \tag{5.17}$$

for the second-kind boundary condition. Also, we have verified that if $\boldsymbol{\beta} = 0$ both solutions agree with the homogeneous half-space Green's function at the transient state (Chang, 1977, Eq. (5.1)).

5.3. An infinite plane

For a two-dimensional plane, in analogous to (4.3), the solution of (5.6) takes the form

$$e^{st'}\overline{G^e} = \frac{1}{2\pi\kappa}K_0(\tilde{\lambda}R). \tag{5.18}$$

Taking the inverse Laplace transform of (5.18) and making use of the identity of (A.7), one finds

$$G^e = \frac{H(t-t')}{\kappa[4\pi(t-t')]}\exp\left[-\frac{R^2}{4\alpha(t-t')} - \alpha\lambda^2(t-t')\right]. \tag{5.19}$$

Thus,

$$G(\mathbf{x},\mathbf{x}';t,t') = \frac{H(t-t')}{\kappa[4\pi(t-t')]}\exp\left(-\frac{R^2}{4\alpha(t-t')}\right)\cdot\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')). \tag{5.20}$$

Letting $\boldsymbol{\beta} = 0$, we recover the known two-dimensional Green's function at transient state for a homogeneous solid (Chang et al., 1973, Eq. (16)).

5.4. A half-plane

Lastly, we consider the transient-state Green's function for a half-plane. The algebraic correspondence of (5.7) suggests that the solutions in the transformed domain, in view of (4.9) and (4.10), should be

$$e^{st'}\overline{G^e} = \frac{1}{2\pi\kappa}[K_0(\tilde{\lambda}R) - K_0(\tilde{\lambda}R_i)], \tag{5.21}$$

for the first kind boundary condition, and

$$e^{st'}\overline{G^e} = \frac{1}{2\pi\kappa}\left[K_0(\tilde{\lambda}R) + K_0(\tilde{\lambda}R_i) + \int_0^\infty h(\zeta)K_0(\tilde{\lambda}R_\zeta)\mathrm{d}\zeta\right], \tag{5.22}$$

for the second kind boundary condition. Now making use of the identity (A.7) together with the basic shifting theorems for Laplace transform, we find that the Green's functions for a graded half-plane take the forms

$$\begin{aligned} G(\mathbf{x},\mathbf{x}';t,t') = {} & \frac{H(t-t')}{4\pi\kappa(t-t')}\exp(-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')) \\ & \times\left(\exp\left(-\frac{R^2}{4\alpha(t-t')}\right) - \exp\left(-\frac{R_i^2}{4\alpha(t-t')}\right)\right), \end{aligned} \tag{5.23}$$

for the first-kind boundary condition, and

$$\begin{aligned} G(\mathbf{x},\mathbf{x}';t,t') = {} & \frac{H(t-t')}{4\pi\kappa(t-t')}\exp[-\boldsymbol{\beta}\cdot(\mathbf{x}+\mathbf{x}') - \alpha\lambda^2(t-t')] \\ & \times\left[\exp\left(-\frac{R^2}{4\alpha(t-t')}\right) + \exp\left(-\frac{R_i^2}{4\alpha(t-t')}\right) + \int_0^\infty h(\zeta)\exp\left(-\frac{R_\zeta^2}{4\alpha(t-t')}\right)\mathrm{d}\zeta\right]. \end{aligned} \tag{5.24}$$

for the second-kind boundary condition. Of course, the variables R, R_i, and R_ς now follow the expressions for a two-dimensional plane, namely Eq. (4.11).

6. Closure

In summary, we have derived a number of Green's functions for conduction phenomena analytically for an exponentially graded solid. Domains of an unbounded space and a half-space, either three-dimensional or two-dimensional, with two types of homogeneous boundary conditions are considered. The exact expressions for these Green's functions are rather compact. Of course, further simplification can be anticipated if we allow that the material possesses further crystallographic symmetries or assume that the grading direction and/or one of the symmetry planes is parallel to the boundary surface. The relative simplicity of the obtained Green's functions offers the prospect for a wide range of applications. For example, they may be employed to study the nonlocal effects of composites and to assess the effective properties of graded random composites. A recent review (Buryachenko, 2001) addressed some of relevant subjects. Further, as in homogeneous materials, we note that the Green's functions also fulfill the symmetry property, $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$ or $G(\mathbf{x},\mathbf{x}';\ t,t') = G(\mathbf{x}',\mathbf{x};\ t,t')$. It should be mentioned that due to the mathematical correspondence between conduction and anti-plane mechanics, the present derived (steady-state) Green's functions apply readily to the context of anti-plane elasticity. Lastly there are some related subjects that one may explore in future studies. For example, one may consider the Green's function associated with a third-kind (convective) boundary condition, the Green's functions for domains of an infinite strip or bi-material, and for materials with cylindrical or spherical anisotropy.

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Appendix A. Some useful identities

We record some useful identities of integral transforms that are employed in the formulation:

$$\mathscr{L}\left\{t^{-\frac{3}{2}}\exp\left(-\frac{a}{t}\right)\right\} = \sqrt{\frac{\pi}{a}}\exp(-2\sqrt{as}), \tag{A.1}$$

$$\mathscr{F}^{-1}\{(\xi^2+\rho^2)^{-\frac{1}{2}}e^{-\sigma(\xi^2+\rho^2)^{\frac{1}{2}}}\} = \frac{1}{\pi}K_0(\rho(x^2+\sigma^2)^{\frac{1}{2}}), \tag{A.2}$$

$$\mathscr{F}^{-1}\{K_0(\sigma(\xi^2+\rho^2)^{\frac{1}{2}})\} = \frac{1}{2}(x^2+\sigma^2)^{-\frac{1}{2}}e^{-\rho(x^2+\sigma^2)^{\frac{1}{2}}}, \tag{A.3}$$

$$\mathscr{F}^{-1}\{\tilde{f}(a\xi)\} = \frac{1}{a}f\left(\frac{x}{a}\right), \tag{A.4}$$

$$\mathscr{F}^{-1}\{\tilde{f}(\xi-p_0)\} = e^{\mathrm{i}p_0x}f(x), \tag{A.5}$$

$$\mathscr{F}^{-1}\{e^{-\mathrm{i}g_0\xi}\tilde{f}(\xi)\} = f(x-g_0), \tag{A.6}$$

$$\mathscr{L}\left\{\frac{1}{2t}\exp\left(-\frac{a^2}{4t}\right)\right\} = K_0(a\sqrt{s}). \tag{A.7}$$

We mention that (A.1) can be found in Sneddon (1972, p. 520); Eqs. (A.2) and (A.3) are, respectively, recorded from page 111 and 125 of Campbell and Foster (1954). Formulae (A.4)–(A.6) are basic shifting theorems of Fourier transforms (Sneddon, 1972, p. 39). Equation (A.7) is recorded from Korn and Korn (1968).

Appendix B. Solution procedure for (3.10)

In this Appendix we outline the solution procedure of (3.10), which will be formulated based on a standard procedure (Hildebrand, 1965, p. 228) for the Green's function of a self-adjoint operator. Here we only derive solution of (3.10) with the boundary condition $(3.11)_1$. The same equation associated with the condition $(3.11)_2$ can be resolved in a similar manner. As a starting point, we first let $u(x_1)$ be a solution of the homogeneous equation $(3.10)_1$ that satisfies the boundary condition $(3.11)_1$ at $x_1 = 0$ and also let $v(x_1)$ be the other homogeneous solution which remains finite as $x_1 \to \infty$. The solution $\widetilde{G}^w$ can be expressed in the forms

$$\widetilde{G}^w(x_1, x_1') = \begin{cases} c_1 u(x_1), & 0 \leqslant x_1 < x_1', \\ c_2 v(x_1), & x_1' < x_1 < \infty, \end{cases} \tag{B.1}$$

in which c_1 and c_2 are some constants and u and v can be determined as

$$\begin{aligned} u(x_1) &= \exp(\mu x_1) - \exp(-\mu x_1), \quad 0 \leqslant x_1 < x_1', \\ v(x_1) &= \exp(-\mu x_1), \quad x_1' < x_1 < \infty. \end{aligned} \tag{B.2}$$

By the method, the unknown coefficients c_1 and c_2 can be determined by demanding that $\widetilde{G}^w$ be continuous at $x_1 = x_1'$,

$$c_2 v(x_1') - c_1 u(x_1') = 0, \tag{B.3}$$

and that the derivative of $\widetilde{G}^w(x_1, x_1')$ with respect to x_1 follows the jump condition

$$c_2 v'(x_1') - c_1 u'(x_1') = -w(x_1')/k_{11}. \tag{B.4}$$

These two conditions, (B.3) and (B.4), will give a unique solution for c_1 and c_2 as

$$c_1 = -\frac{v(x_1')}{\Lambda}, \quad c_2 = -\frac{u(x_1')}{\Lambda}, \qquad \text{with } \Lambda = -\frac{2k_{11}\mu}{w(x_1')}. \tag{B.5}$$

Hence (B.1) takes the form

$$\widetilde{G}^w(x_1, x_1') = -\frac{1}{\Lambda} u(x_{1<}) v(x_{1>}), \tag{B.6}$$

or, equivalently, (3.13), where the variables $x_{1<}$ and $x_{1>}$ have been defined in (3.15).

Appendix C. Fourier inversion of (3.8) with $\widetilde{G}^w$ defined in (3.13)

In this Appendix, we describe detailed derivation procedures for the Fourier inverse of $\widetilde{G}^e$ through (3.8). We recall that the function $\widetilde{G}^e$, defined in (3.8), was obtained from G^w, (3.13), divided by $w(x_1)$, (3.9). To begin with, let us first expand the quantity μ as

$$\mu = (m_{33}\xi_2^2 + 2m_{23}\xi_2\xi_3 + m_{22}\xi_3^2 + \lambda^2 k_{11})^{\frac{1}{2}}/k_{11} = \left[\left(A\left(\xi_2 + \frac{C}{A^2}\xi_3\right)\right)^2 + \left(B^2 - \frac{C^2}{A^2}\right)\xi_3^2 + D^2\right]^{\frac{1}{2}}, \tag{C.1}$$

in which

$$A = \sqrt{m_{33}}/k_{11}, \quad B = \sqrt{m_{22}}/k_{11}, \quad C = m_{23}/k_{11}^2, \quad D^2 = \lambda^2/k_{11}. \tag{C.2}$$

To perform the double Fourier inverse of (3.8), let us consider the first exponential term in the bracket of (3.13). The second term of which can be integrated in a similar manner. To do the inverse transform, we first derive the integral with respect to ξ_2. For clarity, we denote the outcome symbolically as $\mathscr{P}$

$$\begin{aligned} \mathscr{P} &\equiv \mathscr{F}^{-1}\left\{\frac{e^{-\mu(x_{1>}-x_{1<})}}{2\mu k_{11}}\left(\frac{w(x_1')}{w(x_1)}\right)e^{-\mathrm{i}(\xi_2 x_2' + \xi_3 x_3')}; \xi_2 \to x_2\right\} \\ &= \mathscr{F}^{-1}\left\{\frac{e^{-\mu(x_{1>}-x_{1<})}}{2\mu k_{11}} e^{-\mathrm{i}\xi_2\left(x_2' + \frac{k_{12}}{k_{11}}(x_1 - x_1')\right)} e^{-\mathrm{i}\xi_3\left(x_3' + \frac{k_{13}}{k_{11}}(x_1 - x_1')\right)}; \xi_2 \to x_2\right\}. \end{aligned} \tag{C.3}$$

In the above we have incorporated the function $w(x_1')/w(x_1)$ via (3.9) inside the expression. To proceed, we set

$$\begin{aligned} &\xi \to \xi_2, \quad \sigma \to (x_{1>} - x_{1<}), \quad \rho^2 \to \left(B^2 - \frac{C^2}{A^2}\right)\xi_3^2 + D^2, \quad a \to A, \\ &p_0 \to -C\xi_3/A^2, \quad g_0 \to x_2' + k_{12}(x_1 - x_1')/k_{11}, \end{aligned} \tag{C.4}$$

in (A.2) and (A.4)–(A.6). After some manipulations, it can be shown that $\mathscr{P}$ has the form

$$\mathscr{P} = \frac{1}{2\pi k_{11} A} K_0\left((E^2\xi_3^2 + D^2)^{\frac{1}{2}} R_2\right) e^{-\mathrm{i}\xi_3\left[x_3' + \frac{C}{A^2}(x_2 - x_2') + \left(\frac{k_{13}}{k_{11}} - \frac{C k_{12}}{A^2 k_{11}}\right)(x_1 - x_1')\right]}, \tag{C.5}$$

where

$$E^2 = B^2 - \frac{C^2}{A^2}, \quad R_2 = \left[(x_{1>} - x_{1<})^2 + \left(\frac{x_2 - g_0}{A}\right)^2\right]^{\frac{1}{2}}. \tag{C.6}$$

We next invert the function $\mathscr{P}$ with respect to ξ_3 and denote the resulting formula as $\mathscr{Q}(x_1, x_2, x_3)$

$$\mathscr{Q} = \mathscr{F}^{-1}\{\mathscr{P}; \xi_3 \to x_3\}. \tag{C.7}$$

For the Fourier inversion, we now make use of (A.3)–(A.6) by setting that

$$\begin{aligned} &\xi \to \xi_3, \quad \sigma \to R_2, \quad \rho^2 \to D^2, \quad a \to E, \quad p_0 \to 0, \\ &g_0 \to x_3' + \frac{C}{A^2}(x_2 - x_2') + \left(\frac{k_{13}}{k_{11}} - \frac{C}{A^2}\frac{k_{12}}{k_{11}}\right)(x_1 - x_1'). \end{aligned} \tag{C.8}$$

This will allow us to derive (C.7) exactly as

$$\mathscr{Q} = \frac{1}{4\pi k_{11}^{\frac{3}{2}} A E} \frac{\exp(-\lambda R)}{R}, \tag{C.9}$$

where

$$\begin{aligned} R^2 \equiv &\frac{(x_{1>} - x_{1<})^2}{k_{11}} + \frac{1}{k_{11}A^2}\left[-\frac{k_{12}}{k_{11}}(x_1 - x_1') + (x_2 - x_2')\right]^2 \\ &+ \frac{1}{k_{11}E^2}\left[(x_3 - x_3') - \frac{C}{A^2}(x_2 - x_2') - \left(\frac{k_{13}}{k_{11}} - \frac{C}{A^2}\frac{k_{12}}{k_{11}}\right)(x_1 - x_1')\right]^2. \end{aligned} \tag{C.10}$$

The function $\mathcal{Q}$ can be further simplified by noticing that

$$k_{11}^{\frac{3}{2}}AE = k_{11}^{\frac{3}{2}}\frac{m_{33}^{\frac{1}{2}}}{k_{11}}\left(B^2-\frac{C^2}{A^2}\right)^{\frac{1}{2}} = \left[k_{11}m_{33}\left(\frac{m_{22}}{k_{11}^2}-\frac{\frac{m_{23}^2}{k_{11}^4}}{\frac{m_{33}}{k_{11}^2}}\right)\right]^{\frac{1}{2}} = \left[\frac{1}{k_{11}}\left(m_{22}m_{33}-m_{23}^2\right)\right]^{\frac{1}{2}} = (\det\mathbf{k})^{\frac{1}{2}}$$
$$= \kappa. \tag{C.11}$$

Also, in conformity with (C.2) we can rewrite R^2 in the form

$$R^2 = \frac{(x_{1>}-x_{1<})^2}{k_{11}} + \widetilde{R}^2, \tag{C.12}$$

where

$$\widetilde{R}^2 = \frac{k_{11}}{m_{33}}\left[-\frac{k_{12}}{k_{11}}(x_1-x_1') + (x_2-x_2')\right]^2$$
$$+\frac{m_{33}}{\det\mathbf{k}}\left[(x_3-x_3') - \frac{m_{23}}{m_{33}}(x_2-x_2') - \left(\frac{k_{13}}{k_{11}}-\frac{m_{23}}{m_{33}}\frac{k_{12}}{k_{11}}\right)(x_1-x_1')\right]^2. \tag{C.13}$$

It can be proven analytically that the expression R in (C.12) is precisely the one defined in (2.18). Although the form of (C.12) may not be simple, this expression has some merits. Let us take the first exponential term in (3.13) as an example. We see that the term $\exp(x_{1>}-x_{1<})$ only enters the first item in (C.12), while it has no effect on the part $\widetilde{R}$. This suggests that for the second exponential term of (3.13) its inversion can be readily obtained from (C.12) simply by replacing $(x_{1>}-x_{1<})$ with $(x_{1>}+x_{1<})$ without repeating the whole derivations again. In other words, we have

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\mu k_{11}}e^{-\mu(x_{1>}+x_{1<})}\left(\frac{w(x_1')}{w(x_1)}\right)e^{\mathrm{i}\xi_2(x_2-x_2')}e^{\mathrm{i}\xi_3(x_3-x_3')}\mathrm{d}\xi_2\mathrm{d}\xi_3 = \frac{\exp(-\lambda R_{\mathrm{i}})}{4\pi\kappa R_{\mathrm{i}}}, \tag{C.14}$$

where

$$R_i^2 \equiv \frac{(x_{1>}+x_{1<})^2}{k_{11}} + \widetilde{R}^2. \tag{C.15}$$

Hence we have

$$G^e(\mathbf{x},\mathbf{x}') = \frac{\exp(-\lambda R)}{4\pi\kappa R} - \frac{\exp(-\lambda R_{\mathrm{i}})}{4\pi\kappa R_{\mathrm{i}}}. \tag{C.16}$$

Appendix D. Fourier inversion of (3.8) with $\widetilde{G}^w$ given in (3.14)

Here we present the procedures of Fourier inverse transform for (3.8) with $\widetilde{G}^w$ given in (3.14). As a first step, we first rewrite (3.14) as

$$\widetilde{G}^w(x_1,\xi_2,\xi_3) = \frac{w(x_1')}{k_{11}}\left(\frac{1}{2\mu}e^{-\mu(x_{1>}-x_{1<})} + \frac{1}{2\mu}e^{-\mu(x_{1>}+x_{1<})} - \frac{p}{(\mu+p)\mu}e^{-\mu(x_{1>}+x_{1<})}\right). \tag{D.1}$$

The first two terms are exactly identical (apart from a difference in sign) with those in (3.13) and thus their inversions follow immediately from foregoing results derived in Appendix C. Here we only need to consider the third exponential term. For convenience, let us use $\mathcal{K}$ to represent this function. To proceed, we follow an idea of Carslaw (1902) by letting $\mu = \mathrm{i}\eta$, which will give

$$\mathcal{K} \equiv -\frac{p}{(\mu+p)\mu}\exp(-\mu(x_{1>}+x_{1<})) = -\frac{p}{(\mathrm{i}\eta+p)\mathrm{i}\eta}\exp(-\mathrm{i}\eta(x_{1>}+x_{1<}))$$
$$= \left(-\frac{p}{\mathrm{i}\eta}\right)\frac{(p-\mathrm{i}\eta)\cos\eta(x_{1>}+x_{1<})-(\eta+\mathrm{i}p)\sin\eta(x_{1>}+x_{1<})}{p^2+\eta^2}. \tag{D.2}$$

Making use of the following identities of Fourier cosine and sine transforms (Sneddon, 1972)

$$\frac{p}{p^2+\eta^2} = \int_0^\infty \exp(-p\zeta)\cos\eta\zeta\mathrm{d}\zeta, \quad \frac{\eta}{p^2+\eta^2} = \int_0^\infty \exp(-p\zeta)\sin\eta\zeta\mathrm{d}\zeta, \tag{D.3}$$

the function $\mathcal{K}$ is equivalent to

$$\mathcal{K} = -\frac{p}{\mathrm{i}\eta}\int_0^\infty \exp(-p\zeta)\exp(-\mathrm{i}\eta(x_{1>}+x_{1<}+\zeta))\mathrm{d}\zeta = \frac{1}{2\mu}\int_0^\infty h(\zeta)\exp(-\mu(x_{1>}+x_{1<}+\zeta))\mathrm{d}\zeta, \tag{D.4}$$

where

$$h(\zeta) = -2p\ \exp(-p\zeta). \tag{D.5}$$

Up to this stage we have not yet made the Fourier inversion; we simply rewrite the term $\mathcal{K}$ in the form of (D.4). Thus, upon substitution of the third term in (D.1) into (3.8) and (3.4), the Fourier inversion of the corresponding G^e can be expressed as

$$\mathcal{F}^{-1}\left\{\mathcal{F}^{-1}\left\{\frac{1}{2\mu k_{11}}\left(\frac{w(x_1')}{w(x_1)}\right)\int_0^\infty h(\zeta)e^{-\mu(x_{1>}+x_{1<}+\zeta)}\mathrm{d}\zeta e^{-\mathrm{i}(\xi_2x_2'+\xi_3x_3')};\xi_2\to x_2\right\};\xi_3\to x_3\right\}$$
$$= \int_0^\infty h(\zeta)\mathcal{F}^{-1}\left\{\mathcal{F}^{-1}\left\{\frac{1}{2\mu k_{11}}\left(\frac{w(x_1')}{w(x_1)}\right)e^{-\mu(x_{1>}+x_{1<}+\zeta)}e^{-\mathrm{i}(\xi_2x_2'+\xi_3x_3')};\xi_2\to x_2\right\};\xi_3\to x_3\right\}\mathrm{d}\zeta. \tag{D.6}$$

Since ζ is a dummy variable which does not involve in the inversion formulae, we can interchange the integral sign for ζ and the Fourier inversion with respect to ξ_2 and ξ_3. The advantage of this alternative expression of $\mathcal{K}$ is now recognized that the Fourier inversion of (D.6) is directly obtained, without derivations, from (C.15) simply by replacing $(x_{1>}+x_{1<})$ with $(x_{1>}+x_{1<}+\zeta)$. Thus we conclude that the Fourier inversion of the third term in (D.1), incorporating with (3.8) and (3.4), follows

$$\int_0^\infty h(\zeta)\frac{\exp(-\lambda R_\zeta)}{4\pi\kappa R_\zeta}\mathrm{d}\zeta, \quad \text{with } R_\zeta^2 \equiv \frac{(x_{1>}+x_{1<}+\zeta)^2}{k_{11}} + \widetilde{R}^2. \tag{D.7}$$

References

Arfken, G.B., Weber, H.J., 2001. Mathematical Methods for Physicists. Academic Press, San Diego.

Buryachenko, V.A., 2001. Multiparticle effective field and related methods in micromechanics of composite materials. Appl. Mech. Rev. 54, 1–47.

Campbell, G.A., Foster, R.M., 1954. Fourier Integrals for Practical Applications. D. Van Nostrand Company, New York.

Carslaw, H.S., 1902. A problem in conduction of heat. The London, Edinburgh and Philosophical Magazine and Journal of Science, Volume 4, 6th series, 162–165.

Chang, Y.P., Kang, C.S., Chen, D.J., 1973. The use of fundamental Green's functions for the solution of problems of heat conduction in anisotropic media. Int. J. Heat Mass Transfer 16, 1905–1918.

Chang, Y.P., 1977. Analytic solution for heat conduction in anisotropic media in infinite, semi-infinite, and two-plane-bounded regions. Int. J. Heat Mass Transfer 20, 1019–1028.

Chen, T., 1993. Green's functions and the non-uniform transformation problem in a piezoelectric medium. Mech. Res. Commun. 20, 271–278.

DeSanto, J.A., 1992. Scalar Wave Theory—Green's Functions and Applications. Springer-Verlag, New York.

Giannakopoulos, A.E., Suresh, S., 1997. Identation of solids with gradients in elastic properties. I. Point force. Int. J. Solids Struct. 34, 2357–2392.

Gray, L.J., Kaplan, T., Richardson, J.D., Paulino, G.H., 2003. Green's functions and boundary integral analysis for exponentially graded materials: heat conduction. J. Appl. Mech. 70, 543–549.

Greenberg, M.D., 1971. Application of Green's Functions in Science and Engineering. Prentice-Hall, New Jersey.

Hildebrand, F.B., 1965. Methods of Applied Mathematics. Prentice-Hall, New Jersey.

Hirai, T., 1995. Functional gradient materials. Materials Science and Technology: a Comprehensive Treatment, R. J. Brook, ed., Vol. 17B of Processing of Ceramics, Part 2, VCH Verlagsgesellschaft GmbH, Weinheim, Germany, 292-341.

Horgan, C.O., Chan, A.M., 1999. Torsion of functionally graded isotropic linearly elastic bars. J. Elasticity 52, 181–199.

Korn, G.A., Korn, T.M., 1968. Mathematical Handbook for Scientist and Engineers. McGraw-Hill, New York.

Lanczos, C., 1961. Linear Differential Operators. Van Nostrand, New York.

Martin, P.A., Richardson, J.D., Gray, L.J., Berger, J.R., 2002. On Green's function for a three-dimensional exponentially graded elastic solid. Proc. R. Soc. Lond. A 458, 1931–1947.

Milton, G.W., 2002. The Theory of Composites. Cambridge University Press, Cambridge.

Norris, A.N., 1994. Dynamic Green's functions in anisotropic piezoelectric, thermoelastic and poroelastic solids. Proc. R. Soc. Lond. A 447, 175–188.

Reiter, M., Dvorak, G.J., Tvergaard, V., 1997. Micromechanical models for graded composite materials. J. Mech. Phys. Solids. 45, 1281–1302.

Sneddon, I.N., 1972. The Use of Integral Transforms. McGraw-Hill, New York.

Sokolnikoff, I.S., 1956. Mathematical Theory of Elasticity. McGraw-Hill, New York.

Suresh, S., 2001. Graded materials for resistance to contact deformation and damage. Science 292, 2447–2451.

Wang, M., Jasiuk, I., 1998. Effective elastic constants of particulate composites with inhomogeneous interphases. J. Comp. Mater. 32, 1391–1424.

Weng, G.J., 2003. Effective bulk moduli of two functionally graded composites. Acta Mech. 166, 57–67.